DISCRETE MIURA OPERS AND SOLUTIONS OF THE BETHE ANSATZ EQUATIONS

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Abstract. Solutions of the Bethe ansatz equations associated to the XXX model of a simple Lie algebra $\mathfrak{g}$ come in families called the populations. We prove that a population is isomorphic to the flag variety of the Langlands dual Lie algebra ${}^t\mathfrak{g}$. The proof is based on the correspondence between the solutions of the Bethe ansatz equations and special difference operators which we call the discrete Miura opers. The notion of a discrete Miura oper is one of the main results of the paper.

For a discrete Miura oper D, associated to a point of a population, we show that all solutions of the difference equation $DY = 0$ are rational functions, and the solutions can be written explicitly in terms of points composing the population.

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1. Introduction

The Bethe ansatz is a large collection of methods in the theory of quantum integrable models to calculate the spectrum and eigenvectors for a certain commutative subalgebra of observables for an integrable model. This commutative subalgebra includes the Hamiltonian of the model. Its elements are called integrals of motion or conservation laws of the model. Most of recent developments of the Bethe ansatz methods is due to the quantum inverse scattering transform, invented by the Leningrad school of mathematical physics. The bibliography on the Bethe ansatz method is enormous. We refer the reader to reviews [BIK, Fa, FT].

In the theory of the Bethe ansatz one assigns the Bethe ansatz equations to an integrable model. Then a solution of the Bethe ansatz equations gives an eigenvector of commuting Hamiltonians of the model. The general conjecture is that the constructed vectors form a basis in the space of states of the model. The first step to that conjecture

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is to count the number of solutions of the Bethe ansatz equations. One can expect that the number of solutions is equal to the dimension of the space of states of the model.

The Bethe ansatz equations of the XXX model is a system of algebraic equations associated to a Kac-Moody algebra $\mathfrak{g}$, a non-zero step $h \in \mathbb{C}$, complex numbers $z_1, \dots, z_n$, integral dominant $\mathfrak{g}$-weights $\Lambda_1, \dots, \Lambda_n$ and an integral $\mathfrak{g}$-weight Λ_∞, see [OW] and Section 2.2 below.

To approach the corresponding counting problem, to every solution of the Bethe ansatz equations we assign an object called the population of solutions. We expect that it would be easier to count populations than individual solutions. For instance, if the Kac-Moody algebra is of type A_r, then each population corresponds to a point of the intersection of suitable Schubert varieties in a suitable Grassmannian variety, as was shown in [MV3]. Then the Schubert calculus allows us to count the number of intersection points of the Schubert varieties and to give an upper bound on the number of populations, see [MV1] and [MV3].

A population of solutions is an interesting object. It is an algebraic variety. It is finite-dimensional, if the Weyl group of the Kac-Moody algebra is finite. In this paper we prove that a population is isomorphic to the flag variety of the Langlands dual Lie algebra ${}^t\mathfrak{g}$. The proof is based on the correspondence between solutions of the Bethe ansatz equations and special difference operators which we call the discrete Miura opers.

Let G be the complex simply connected group with Lie algebra ${}^t\mathfrak{g}$. To every solution $\boldsymbol{t}$ of the Bethe ansatz equations we assign a linear difference operator $D_{\boldsymbol{t}} = \partial_h - V_{\boldsymbol{t}}$ where $\partial_h : f(x) \to f(x+h)$ is the shift operator and $V_{\boldsymbol{t}}(x)$ is a suitable rational G-valued function. We call that difference operator a discrete Miura oper. Our discrete Miura opers are analogs of the differential operators considered by V. Drinfeld and V. Sokolov in their study of the KdV type equations [DS].

Different solutions of the Bethe ansatz equations correspond to different discrete Miura opers. The discrete Miura opers, corresponding to points of a given population, form an equivalence class with respect to a suitable gauge equivalence. Thus a population is isomorphic to an equivalence class of discrete Miura opers. We show that an equivalence class of discrete Miura opers is isomorphic to the flag variety of ${}^t\mathfrak{g}$.

If $D_{\boldsymbol{t}}$ is the discrete Miura oper corresponding to a solution $\boldsymbol{t}$ of the Bethe ansatz equations, then the set of solutions of the difference equation $D_{\boldsymbol{t}} Y = 0$ with values in a suitable space is an important characteristics of $\boldsymbol{t}$. It turns out that, for any simple Lie algebra and any solution $\boldsymbol{t}$ of the Bethe ansatz equations, the difference equation $D_{\boldsymbol{t}} Y = 0$ has a rational fundamental matrix of solutions. Moreover, all solutions of the difference equation $D_{\boldsymbol{t}} Y = 0$ can be written explicitly in terms of points composing the population, originated at $\boldsymbol{t}$. Thus, the population of solutions of the Bethe ansatz equations "solves" the Miura difference equation $D_{\boldsymbol{t}} Y = 0$ in rational functions. This is the second main result of the paper.

The results of this paper in the cases of A_r and B_r were first obtained in [MV3].

The populations related to the Gaudin model of a Kac-Moody algebra $\mathfrak{g}$ were introduced in [MV1]. In [MV1] we conjectured that every $\mathfrak{g}$-population is isomorphic to the flag variety of the Langlands dual Kac-Moody algebra $\mathfrak{g}^L$. That conjecture was proved for $\mathfrak{g}$ of type A_r, B_r, C_r in [MV2], for $\mathfrak{g}$ of type G_2 in [BM], for all simple Lie algebras in [F1] and [MV4]. The ideas of [MV4] motivated the present paper.

There are different versions of the XXX Bethe ansatz equations associated to a simple Lie algebra, see [OW, MV2, MV3]. Ogievetsky and Wiegman introduced in [OW] a set of Bethe ansatz equations for any simple Lie algebra $\mathfrak{g}$. For $\mathfrak{g}$ of type A_r, D_r, E_6, E_7, E_8 the Ogievetsky-Wiegman equations are the Bethe ansatz equations considered in this paper. For other simple Lie algebras the Ogievetsky-Wiegman equations are different from the Bethe ansatz equations considered in this paper, see Section 2.6.

The discrete Miura opers, considered in this paper, are discrete versions of the special differential operators called the Miura opers and introduced in [DS]. The Miura opers play an essential role in the Drinfeld-Sokolov reduction and geometric Langlands correspondence, see [DS, FFR, F2, FRS]. It would be interesting to see if our discrete opers may play a similar role in discrete versions of the Drinfeld-Sokolov reduction and geometric Langlands correspondence.

Considerations of the present paper are in the spirit of the geometric Langlands correspondence. Namely, we start from a solution $\boldsymbol{t}$ of the Bethe ansatz equations associated to a simple Lie algebra $\mathfrak{g}$, that is we start from a Bethe eigenvector of the commuting Hamiltonians of the XXX $\mathfrak{g}$ model. Having a solution $\boldsymbol{t}$ we construct the associated discrete Miura G-oper $D_{\boldsymbol{t}}$, whose fundamental matrix is a rational function. Having $D_{\boldsymbol{t}}$ we may recover the $\mathfrak{g}$ population of solutions of the Bethe ansatz equations originated at $\boldsymbol{t}$. In particular we may recover $\boldsymbol{t}$ and the associated Bethe eigenvector of the commuting Hamiltonians.

The fact that the discrete Miura oper has a rational fundamental matrix of solutions is a discrete analog of the fact that a differential Miura oper has the trivial monodromy group, see [F1, MV4].

The paper is organized as follows. In Section 2 we introduce populations of solutions of the Bethe ansatz equations. In Section 3 we discuss elementary properties of discrete Miura opers corresponding to solutions of the Bethe ansatz equations. In Section 4 we give explicit formulas for solutions of the difference equation $D_{\boldsymbol{t}} Y = 0$, see Theorems 4.1, 4.2. In Section 5 we prove that the variety of gauge equivalent marked discrete Miura $\mathfrak{g}$ opers is isomorphic to the flag variety of ${}^t\mathfrak{g}$, see Theorem 5.1. We discuss the relations between the Bruhat cell decomposition of the flag variety and the populations of solutions of the Bethe ansatz equations in Section 6. The main results of the paper are Corollaries 6.5 and 6.6.

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2. Populations of Solutions of the Bethe Equations, [MV2]

2.1. **Kac-Moody algebras.** Let $A = (a_{i,j})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{i,i} = 2$, $a_{i,j} = 0$ if and only $a_{j,i} = 0$, $a_{i,j} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$. We assume that A is symmetrizable, i.e. there exists a diagonal matrix $D = \mathrm{diag}\{d_1, \dots, d_r\}$ with positive integers d_i such that $B = DA$ is symmetric.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see [K], §1.2), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where d is the dimension of the kernel of the Cartan matrix A.

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \dots, r$, be the sets of simple roots, coroots, respectively. We have

$$\begin{aligned} (\alpha_i, \alpha_j) &= d_i\, a_{i,j}, \\ \langle \lambda, \alpha_i^\vee \rangle &= 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i), \qquad \lambda \in \mathfrak{h}^*. \end{aligned}$$

In particular, $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \,|\, \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* \,|\, \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$ be the sets of integral and dominant integral weights.

Fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \dots, r$. We have $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$.

The Weyl group $\mathcal{W} \in \mathrm{End}(\mathfrak{h}^*)$ is generated by reflections s_i, $i = 1, \dots, r$,

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \qquad \lambda \in \mathfrak{h}^*.$$

We use the notation

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \qquad w \in \mathcal{W},\ \lambda \in \mathfrak{h}^*,$$

for the shifted action of the Weyl group.

The Kac-Moody algebra $\mathfrak{g}(A)$ is generated by $\mathfrak{h}$, $e_1, \dots, e_r, f_1, \dots, f_r$ with defining relations

$$\begin{aligned} [e_i, f_j] &= \delta_{i,j}\, \alpha_i^\vee, \qquad i, j = 1, \dots r, \\ [h, h'] &= 0, \qquad h, h' \in \mathfrak{h}, \\ [h, e_i] &= \langle \alpha_i, h \rangle\, e_i, \qquad h \in \mathfrak{h},\ i = 1, \dots r, \\ [h, f_i] &= -\langle \alpha_i, h \rangle\, f_i, \qquad h \in \mathfrak{h},\ i = 1, \dots r, \end{aligned}$$

and the Serre's relations

$$(\mathrm{ad}\, e_i)^{1-a_{i,j}}\, e_j = 0, \qquad (\mathrm{ad}\, f_i)^{1-a_{i,j}}\, f_j = 0,$$

for all $i \neq j$. The generators $\mathfrak{h}$, $e_1, \dots, e_r, f_1, \dots, f_r$ are called the Chevalley generators.

Denote $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra generated by $e_1, \dots, e_r$ (resp. $f_1, \dots, f_r$). Then $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$. Set $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$.

The Kac-Moody algebra ${}^t\mathfrak{g} = \mathfrak{g}({}^tA)$ corresponding to the transposed Cartan matrix ${}^tA$ is called *Langlands dual* to $\mathfrak{g}$. Let ${}^t\alpha_i \in {}^t\mathfrak{h}^*$, ${}^t\alpha_i^\vee \in {}^t\mathfrak{h}$, $i = 1, \dots, r$, be the sets of simple roots, coroots of ${}^t\mathfrak{g}$, respectively. Then

$$\langle {}^t\alpha_i, {}^t\alpha_j^\vee \rangle = \langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$$

for all i, j.

2.2. **The Bethe ansatz equations with parameters $\boldsymbol{b}$.** Fix a Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, a non-negative integer n, a collection of dominant integral weights $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$, $\Lambda_i \in \mathcal{P}^+$, and complex numbers $\boldsymbol{z} = (z_1, \dots, z_n)$. Fix a non-zero complex number h.

Fix a collection $\boldsymbol{b} = (b_{i,m})_{i,m=1,\ i\neq m}^r$ of complex numbers. We say that the parameters $\boldsymbol{b}$ are *symmetric* if they satisfy the condition:

$$b_{i,m} \;+\; b_{m,i} \;=\; h \tag{1}$$

for all $i, m \in \{1, \dots, r\},\ i \neq m$.

Choose a collection of non-negative integers $\boldsymbol{l} = (l_1, \dots, l_r) \in \mathbb{Z}_{\geq 0}^r$. The choice of $\boldsymbol{l}$ is equivalent to the choice of the weight

$$\Lambda_\infty \;= \sum_{i=1}^{n} \Lambda_i - \sum_{j=1}^{r} l_j \alpha_j \in \mathcal{P}.$$

The weight Λ_∞ will be called *the weight at infinity.* Set $\bar{\boldsymbol{\Lambda}} = (\Lambda_1, \dots, \Lambda_n, \Lambda_\infty)$. Let

$$\boldsymbol{t} \;=\; \{\ t_j^{(i)} \in \mathbb{C} \mid i = 1, \dots, r,\ j = 1, \dots, l_i\ \}$$

be a collection of complex numbers.

The XXX *Bethe ansatz equations associated to* $\bar{\boldsymbol{\Lambda}}$, $\boldsymbol{z}$, $\boldsymbol{b}$ is the following system of algebraic equations with respect to the variables $\boldsymbol{t}$:

$$\prod_{s=1}^{n} \frac{t_j^{(i)} \;-\; z_s \;+\; (\Lambda_s, \alpha_i^\vee)\, h/2}{t_j^{(i)} \;-\; z_s \;-\; (\Lambda_s, \alpha_i^\vee)\, h/2} \;\times \tag{2}$$
$$\prod_{m=1,\ m\neq i}^{r} \left(\prod_{k=1}^{l_m} \frac{t_j^{(i)} \;-\; t_k^{(m)} \;+\; b_{i,m}}{t_j^{(i)} \;-\; t_k^{(m)} \;+\; b_{i,m} \;-\; h} \right)^{-a_{i,m}} \prod_{k=1,\ k\neq j}^{l_i} \frac{t_j^{(i)} \;-\; t_k^{(i)} \;-\; h}{t_j^{(i)} \;-\; t_k^{(i)} \;+\; h} \;=\; 1\ ,$$

where $i = 1, \dots, r$, $j = 1, \dots, l_i$.

The product of symmetric groups $S_{\boldsymbol{l}} = S_{l_1} \times \cdots \times S_{l_r}$ acts on the set of solutions of (2) permuting the coordinates with the same upper index.

For $i = 1, \dots, r$, consider the l_i equations (2) with fixed upper index i. We call that system of equations *the Bethe ansatz equations with fixed upper index* i.

2.3. **Polynomials representing solutions of the Bethe ansatz equations.** For a given $\boldsymbol{t} = (t_j^{(i)})$ introduce an r-tuple of polynomials $\boldsymbol{y} = (y_1(x), \dots, y_r(x))$, where

$$y_i(x) \;=\; \prod_{j=1}^{l_i} (x - t_j^{(i)}). \tag{3}$$

Each polynomial is considered up to multiplication by a non-zero number. The tuple defines a point in the direct product $\mathbb{P}(\mathbb{C}[x])^r$ of r copies of the projective space associated to the vector space of polynomials in x. We say that *the tuple* $\boldsymbol{y}$ *represents the collection of numbers* $\boldsymbol{t}$.

It is convenient to think that if a polynomial y_k of a tuple $\boldsymbol{y} = (y_1, \dots, y_r) \in \mathbb{P}(\mathbb{C}[x])^r$ has degree zero, then it means that the collection $\boldsymbol{t} = (t_j^{(i)})$ has no $t_j^{(i)}$-s with $i = k$.

For $i = 1, \dots, r$ introduce polynomials

$$T_i(x) \; = \; \prod_{s=1}^{n} \prod_{p=0}^{(\Lambda_s, \alpha_i^\vee)-1} (x \; - \; z_s \; + \; (\Lambda_s, \alpha_i^\vee) \; h/2 \; - \; ph) \tag{4}$$

and

$$Q_i(x) \; = \; T_i(x) \prod_{m=1,\; m\neq i}^{r} y_m(x + b_{i,m})^{-a_{i,m}} \; . \tag{5}$$

We say that the tuple $\boldsymbol{y}$ is *generic with respect to weights* $\boldsymbol{\Lambda}$*, numbers* $\boldsymbol{z}$*, and parameters* $\boldsymbol{b}$ if for every $i = 1, \dots, r$ the polynomial $y_i(x)$ has no multiple roots and no common roots with polynomials $y_i(x \; + \; h)$, and $Q_i(x)$.

If $\boldsymbol{y}$ represents a solution $\boldsymbol{t}$ of the Bethe ansatz equations (2) and $\boldsymbol{y}$ is generic, then the $S_{\boldsymbol{l}}$-orbit of $\boldsymbol{t}$ is called *a Bethe solution of (2).*

The Bethe ansatz equations can be written as

$$\frac{Q_i(\, t_j^{(i)} \,)}{Q_i(\, t_j^{(i)} \; - \; h\,)} \prod_{k=1,\; k\neq j}^{l_i} \frac{t_j^{(i)} \; - \; t_k^{(i)} \; - \; h}{t_j^{(i)} \; - \; t_k^{(i)} \; + \; h} \; = \; 1 \; , \tag{6}$$

where $i = 1, \dots, r$, $j = 1, \dots, l_i$.

For $i = 1, \dots, r$, a tuple $\boldsymbol{y}$ is called *fertile in the* i*-th direction with respect to* $\boldsymbol{\Lambda}, \; \boldsymbol{z}, \; \boldsymbol{b}$, if there exists a polynomial $\tilde{y}_i$ satisfying the equation

$$y_i(x+h) \; \tilde{y}_i(x) \; - \; y_i(x) \; \tilde{y}_i(x+h) \; = \; Q_i(x) \; . \tag{7}$$

A tuple $\boldsymbol{y}$ is called *fertile with respect to* $\boldsymbol{\Lambda}, \; \boldsymbol{z}, \; \boldsymbol{b}$, if it is fertile in all directions $i = 1, \dots, r$.

Example. The tuple $(1, \dots, 1)$ is fertile with respect to any given $\boldsymbol{\Lambda}, \; \boldsymbol{z}, \; \boldsymbol{b}$.

Instead of saying that $\boldsymbol{y}$ is generic or fertile with respect to $\boldsymbol{\Lambda}, \; \boldsymbol{z}, \; \boldsymbol{b}$ we will also say that $\boldsymbol{y}$ is *generic or fertile with respect to polynomials* $T_1(x), \dots, T_r(x)$ *and parameters* $\boldsymbol{b}$.

If $\boldsymbol{y}$ is fertile in the i-th direction and $\tilde{y}_i$ is a polynomial solution of (7), then the tuple

$$\boldsymbol{y}^{(i)} \; = \; (y_1, \dots, \tilde{y}_i, \dots, y_r) \; \in \; \mathbb{P}(\mathbb{C}[x])^r \tag{8}$$

is called *an immediate descendant* of $\boldsymbol{y}$ in the i-th direction.

If $\tilde{y}_i$ is a solution of (7), then $\tilde{y}_i + cy_i$ is a solution too for any $c \in \mathbb{C}$.

Lemma 2.1 ([MV1, MV2])**.** *Assume that* $\boldsymbol{y}$ *is generic. Let* $\tilde{y}_i$ *be a solution of (7). Then the tuple* $(y_1, \dots, \tilde{y}_i + cy_i, \dots, y_r)$ *is generic for almost all* $c \in \mathbb{C}$*. The exceptions form a finite subset in* $\mathbb{C}$*.* □

Lemma 2.2 ([MV1, MV2])**.** *Let $\boldsymbol{y}^j$, $j = 1, 2, \dots,$ be a sequence of tuples in $\mathbb{P}(\mathbb{C}[x])^r$ which has a limit $\boldsymbol{y}$. Assume that all tuples $\boldsymbol{y}^j$ are fertile. Then $\boldsymbol{y}$ is fertile.* □

Lemma 2.3 (see [MV1, MV2])**.** *Denote $\tilde{l}_i = \deg \tilde{y}_i$ and $\Lambda_\infty^{(i)} = \sum_{s=1}^n \Lambda_s - \tilde{l}_i\alpha_i - \sum_{j=1,\ j\neq i}^r l_j\alpha_j$. If $\tilde{l}_i \neq l_i$, then*

$$\Lambda_\infty^{(i)} \;=\; s_i \cdot \Lambda_\infty \;,$$

where $s_i\cdot$ is the shifted action of the i-th reflection of the Weyl group. □

Theorem 2.4 (cf. [MV2])**.**

(i) *Let a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ be generic. Let $i \in \{1, \dots, r\}$. Then $\boldsymbol{y}$ is fertile in the i-th direction if and only if $\boldsymbol{t}$ satisfies the Bethe ansatz equations (2) with fixed upper index i.*

(ii) *Let parameters $\boldsymbol{b}$ be symmetric. Let $\boldsymbol{y}$ be generic and fertile. Let $i \in \{1, \dots, r\}$. Let $\boldsymbol{y}^{(i)}$ be an immediate descendant of $\boldsymbol{y}$ in the i-th direction. Assume that $\boldsymbol{y}^{(i)}$ is generic. Then $\boldsymbol{y}^{(i)}$ is fertile.*

Proof. To prove (i) introduce $g(x) = \tilde{y}_i(x)/y_i(x)$ and write (7) as

$$g(x) \;-\; g(x+h) \;=\; \frac{Q_i(x)}{y_i(x)\; y_i(x+h)}\;. \tag{9}$$

The tuple $\boldsymbol{y}$ is fertile in the i-th direction if and only if there exists a rational function $g(x)$ satisfying (9). A function $g(x)$ exists if and only if

$$\operatorname{Res}_{x=t_j^{(i)}} \frac{Q_i(x)}{y_i(x)\; y_i(x+h)} \;=\; -\operatorname{Res}_{x=t_j^{(i)}-h} \frac{Q_i(x)}{y_i(x)\; y_i(x+h)} \tag{10}$$

for $j = 1, \dots, l_j$. The systems of equations (10) is equivalent to the system of the Bethe ansatz equations with fixed upper index i. Part (i) is proved.

To prove (ii) we check that the Bethe ansatz equations (2) are satisfied for roots of polynomials composing the tuple $\boldsymbol{y}^{(i)}$.

The Bethe ansatz equations with upper index i are satisfied for roots of $\boldsymbol{y}^{(i)}$ according to part (i).

If m is such that $a_{i,m} = 0$, then the Bethe ansatz equations with upper index m for roots of $\boldsymbol{y}^{(i)}$ are the same as for $\boldsymbol{y}$, since the roots of $\tilde{y}_i$ do not enter those equations.

Let m be such that $a_{i,m} \neq 0$. Write (7) as

$$\frac{y_i(x+h)}{y_i(x)} \;-\; \frac{\tilde{y}_i(x+h)}{\tilde{y}_i(x)} \;=\; \frac{Q_i(x)}{\tilde{y}_i(x)\; y_i(x)}\;.$$

Substitute to this equation the zeros of the polynomial $y_m(x + b_{i,m})$ and get

$$\frac{y_i(t_k^{(m)} - b_{i,m} + h)}{y_i(t_k^{(m)} - b_{i,m})} \;=\; \frac{\tilde{y}_i(t_k^{(m)} - b_{i,m} + h)}{\tilde{y}_i(t_k^{(m)} - b_{i,m})} \tag{11}$$

for $k = 1, \dots, l_m$.

The Bethe ansatz equations with upper index m for roots of $\boldsymbol{y}^{(i)}$ contain the factor $\tilde{y}_i(t_k^{(m)}+b_{m,i})/\tilde{y}_i(t_k^{(m)}+b_{m,i}-h)$ while the Bethe ansatz equations with upper index m for roots of $\boldsymbol{y}$ contain the factor $y_i(t_k^{(m)}+b_{m,i})/y_i(t_k^{(m)}+b_{m,i}-h)$. By (11) the two ratios are equal if parameters $\boldsymbol{b}$ are symmetric. Hence the Bethe ansatz equations with upper index m are satisfied for roots of $\boldsymbol{y}^{(i)}$ if they are satisfied for roots of $\boldsymbol{y}$. Part (ii) is proved. □

2.4. **Simple reproduction procedure.** Assume that parameters $\boldsymbol{b}$ are symmetric.

Let $\boldsymbol{y}$ represent a Bethe solution of (2). Let $i \in \{1,\dots,r\}$, and let $\tilde{y}_i$ be a polynomial solution of equation (7). For complex numbers c_1 and c_2, not both equal to zero, consider the tuple

$$\boldsymbol{y}^{(i)}_{(c_1:c_2)} \;=\; (y_1,\dots,\; c_1\tilde{y}_i+c_2y,\;\dots,\; y_r) \quad \in \mathbb{P}(\mathbb{C}[x])^r\ .$$

The tuples form a one-parameter family. The parameter space of the family is the projective line $\mathbb{P}^1$ with projective coordinates $(c_1:c_2)$. We have a map

$$Y_{\boldsymbol{y},i}\;:\;\mathbb{P}^1\;\to\mathbb{P}(\mathbb{C}[x])^r\ ,\qquad (c_1:c_2)\;\mapsto\;\boldsymbol{y}^{(i)}_{(c_1:c_2)}\ .$$

Almost all tuples $\boldsymbol{y}^{(i)}_{(c_1:c_2)}$ are generic. The exceptions form a finite set in $\mathbb{P}^1$.

Thus, starting with a tuple $\boldsymbol{y}$, representing a Bethe solution of equations (2) associated to numbers $z_1,\dots,z_n$, integral dominant weights $\Lambda_1,\dots,\Lambda_n$, a weight Λ_∞ at infinity, parameters $\boldsymbol{b}$, and an index $i\in\{1,\dots,r\}$, we construct a family $Y_{\boldsymbol{y},i}:\mathbb{P}^1\to\mathbb{P}(\mathbb{C}[x])^r$ of fertile tuples. For almost all $c\in\mathbb{P}^1$ (with finitely many exceptions only), the tuple $Y_{\boldsymbol{y},i}(c)$ represents a Bethe solution of the Bethe ansatz equations associated to points $z_1,\dots,z_n$, integral dominant weights $\Lambda_1,\dots,\Lambda_n$, parameters $\boldsymbol{b}$, and a suitable weight at infinity.

We call this construction *the simple reproduction procedure in the i-th direction.*

2.5. **General reproduction procedure.** Assume that parameters $\boldsymbol{b}$ are symmetric.

Assume that a tuple $\boldsymbol{y}\in\mathbb{P}(\mathbb{C}[x])^r$ represents a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z},\ \bar{\boldsymbol{\Lambda}},\ \boldsymbol{b}$.

Let $\boldsymbol{i}=(i_1,\dots,i_k)$, $1\le i_j\le r$, be a sequence of natural numbers. We define a k-parameter family of fertile tuples

$$Y_{\boldsymbol{y},\boldsymbol{i}}\;:\;(\mathbb{P}^1)^k\;\to\mathbb{P}(\mathbb{C}[x])^r$$

by induction on k, starting at $\boldsymbol{y}$ and successively applying the simple reproduction procedure in directions $i_1,\dots,i_k$. The image of this map is denoted by $P_{\boldsymbol{y},\boldsymbol{i}}$.

For a given $\boldsymbol{i}=(i_1,\dots,i_k)$, almost all tuples $Y_{\boldsymbol{y},\boldsymbol{i}}(\boldsymbol{c})$ represent Bethe solutions of the Bethe ansatz equations associated to points $z_1,\dots,z_n$, dominant integral weights $\Lambda_1,\dots,\Lambda_n$, symmetric parameters $\boldsymbol{b}$, and suitable weights at infinity. Exceptional values of $c\in(\mathbb{P}^1)^k$ are contained in a proper algebraic subset.

It is easy to see that if $\boldsymbol{i}' = (i'_1, \dots, i'_{k'}),\ 1 \le i'_j \le r$, is a sequence of natural numbers, and the sequence $\boldsymbol{i}'$ is contained in the sequence $\boldsymbol{i}$ as an ordered subset, then $P_{\boldsymbol{y},\boldsymbol{i}'}$ is a subset of $P_{\boldsymbol{y},\boldsymbol{i}}$.

The union

$$P_{\boldsymbol{y}} \;=\; \cup_{\boldsymbol{i}}\, P_{\boldsymbol{y},\boldsymbol{i}} \;\subset \mathbb{P}(\mathbb{C}[x])^r \ ,$$

where the summation is over all of sequences $\boldsymbol{i}$, is called *the population of solutions of the Bethe ansatz equations associated* to the Kac-Moody algebra $\mathfrak{g}$, integral dominant weights $\Lambda_1, \dots, \Lambda_n$, numbers $z_1, \dots, z_n$, symmetric parameters $\boldsymbol{b}$, and *originated* at y.

If two populations with the same $\boldsymbol{\Lambda},\ \boldsymbol{z},\ \boldsymbol{b}$ intersect, then they coincide.

If the Weyl group is finite, then all tuples of a population consist of polynomials of bounded degree. Thus, if the Weyl group of $\mathfrak{g}$ is finite, then a population is an irreducible projective variety.

Every population P has a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$, $\deg y_i = l_i$, such that the weight $\Lambda_\infty = \sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i\alpha_i$ is dominant integral, see [MV1, MV2].

Conjecture 2.1 ([MV2])**.** *Every population, associated to a Kac-Moody algebra* $\mathfrak{g}$ *, dominant integral weights* $\Lambda_1, \dots, \Lambda_n$*, points* $z_1, \dots, z_n$*, symmetric parameters* $\boldsymbol{b}$*, is an algebraic variety isomorphic to the flag variety associated to the Kac-Moody algebra* ${}^t\mathfrak{g}$ *which is Langlands dual to* $\mathfrak{g}$ *. Moreover, the parts of the family corresponding to tuples of polynomials with fixed degrees are isomorphic to Bruhat cells of the flag variety.*

The conjecture is proved for the Lie algebra of type A_r in [MV3]. In this paper we prove the conjecture for every simple Lie algebra.

2.6. **Special symmetric parameters $\boldsymbol{b}$.** Here are two examples of symmetric parameters $\boldsymbol{b}$.

The parameters $\boldsymbol{b}$ are symmetric if

$$b_{i,m} \;= \frac{h}{2} \qquad \text{for all } i \ne m \ . \tag{12}$$

The parameters $\boldsymbol{b}$ satisfying (12) will be called *the Ogievetsky-Wiegman parameters*, cf. [OW, MV3].

The parameters $\boldsymbol{b}$ are symmetric if

$$b_{i,m} \;=\; 0 \;\text{ for } i > m \qquad \text{and} \qquad b_{i,m} \;=\; h \;\text{ for } i < m \ . \tag{13}$$

The parameters $\boldsymbol{b}$ satisfying (13) will be called *the special symmetric parameters*, cf. [OW, MV2].

Let $\boldsymbol{b}^1 = (b^1_{i,m})^r_{i,m=1,\ i\ne m}$ and $\boldsymbol{b}^2 = (b^2_{i,m})^r_{i,m=1,\ i\ne m}$ be two collections of parameters. We say that $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ are *gauge equivalent* if there exist complex numbers $d^{(1)}, \dots, d^{(r)}$ with the following property. We require that for any tuple $(y_1(x), \dots, y_r(x))$, fertile respect to some polynomials $T_1(x), \dots, T_r(x)$ and parameters $\boldsymbol{b}^1$, the tuple $(y_1(x + d^{(1)}), \dots, y_r(x + d^{(r)}))$ is fertile with respect to polynomials $T_1(x + d^{(1)}), \dots, T_r(x + d^{(r)})$ and parameters $\boldsymbol{b}^2$.

If $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ are gauge equivalent and P is a population associated to some polynomials $T_1(x), \dots, T_r(x)$ and parameters $\boldsymbol{b}^1$, then the set

$$\{(y_1(x+d^{(1)}), \dots, y_r(x+d^{(r)})) \mid (y_1(x), \dots, y_r(x)) \in P\}$$

is a population associated to polynomials $T_1(x+d^{(1)}), \dots, T_r(x+d^{(r)})$ and parameters $\boldsymbol{b}^2$.

Theorem 2.5. *Let $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ be symmetric parameters. Assume that the Dynkin diagram of the Cartan matrix A of the Lie algebra $\mathfrak{g}$ is a tree. Then $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ are gauge equivalent.*

Proof. We will introduce parameters $\boldsymbol{b}^3 = (b^3_{i,j})$ in terms of the Dynkin diagram and will prove that $\boldsymbol{b}^1$ and $\boldsymbol{b}^3$ are equivalent. That will prove the theorem.

Let $v_1, \dots, v_r$ be vertices of the Dynkin diagram corresponding to the roots $\alpha_1, \dots, \alpha_r$, respectively. For $i = 2, \dots, r$, let $v_{i_1}, \dots, v_{i_k}$ be the unique sequence of distinct vertices of the Dynkin diagram such that for $j = 1, \dots, k-1$ the vertices v_{i_j} and $v_{i_{j+1}}$ are connected by an edge, and $v_{i_1} = v_1$, $v_{i_k} = v_i$. The number k will be called the distance between v_1 and v_i and denoted by δ_i. Let $\boldsymbol{b}^3$ be defined by the rule:

$$b^3_{i,j} = 0 \quad \text{if} \quad \delta_i > \delta_j, \qquad b^3_{i,j} = h \quad \text{if} \quad \delta_i < \delta_j,$$

$$b^3_{i,j} = 0 \quad \text{if} \quad \delta_i = \delta_j \quad \text{and} \quad i > j, \qquad b^3_{i,j} = h \quad \text{if} \quad \delta_i = \delta_j \quad \text{and} \quad i < j.$$

Clearly $\boldsymbol{b}^3$ is symmetric.

Define $(d^{(1)}, \dots, d^{(r)})$. Set

$$d^{(i)} \; = \; b^1_{i_1,i_2} \; + \; b^1_{i_2,i_3} \; + \; \dots \; + \; b^1_{i_{k-1},i_k} \; - \; (k-1)h$$

for $i > 1$ and set $d^{(1)} = 0$. It is easy to see that the sequence $(d^{(1)}, \dots, d^{(r)})$ establishes the equivalence of $\boldsymbol{b}^1$ and $\boldsymbol{b}^3$. □

Corollary 2.6. *If the Dynkin diagram is a tree, then any set of symmetric parameters is gauge equivalent to the set of special symmetric parameters given by (13).*

Corollary 2.7. *If $\mathfrak{g}$ is simple, then any set of symmetric parameters is gauge equivalent to the set of special symmetric parameters given by (13).*

Ogievetsky and Wiegman considered in [OW] a set of Bethe ansatz equations for any simple Lie agebra $\mathfrak{g}$. For $\mathfrak{g}$ of type A_r, D_r, E_6, E_7, E_8 the Ogievetsky-Wiegman equations are the Bethe ansatz equations associated to parameters given by (12). For other simple Lie algebras the Ogievetsky-Wiegman equations are different from the Bethe ansatz equations considered in this paper.

For $\mathfrak{g}$ of type A_r we considered in [MV3] the Bethe ansatz equations associated to the special symmetric parameters given by (13).

2.7. **Diagonal sequences of polynomials associated to a Bethe solution and a sequence of indices.** In this section we assume that parameters $\boldsymbol{b}$ are symmetric. We introduce notions which will be used in Chapter 4 to construct solutions of difference equations.

Lemma 2.8. *Assume that a tuple $\boldsymbol{y} \in \mathbb{P}(\mathbb{C}[x])^r$ represents a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z}$, $\bar{\boldsymbol{\Lambda}}$, symmetric parameters $\boldsymbol{b}$. Let $\boldsymbol{i} = (i_1, \dots, i_k)$, $1 \le i_j \le r$, be a sequence of natural numbers. Then there exist tuples $\boldsymbol{y}^{(i_1)} = (y_1^{(i_1)}, \dots, y_r^{(i_1)})$, $\boldsymbol{y}^{(i_1,i_2)} = (y_1^{(i_1,i_2)}, \dots, y_r^{(i_1,i_2)})$, $\dots$, $\boldsymbol{y}^{(i_1,\dots,i_k)} = (y_1^{(i_1,\dots,i_k)}, \dots, y_r^{(i_1,\dots,i_k)})$ in $\mathbb{P}(\mathbb{C}[x])^r$ such that*

(i)
$$y_{i_1}^{(i_1)}(x)\ y_{i_1}(x+h) \ - \ y_{i_1}^{(i_1)}(x+h)\ y_{i_1}(x) = T_{i_1}(x) \prod_{m=1,\ m\neq i_1}^{r} (y_m(\,x+b_{i_1,m}\,))^{-a_{i_1,m}}$$

and $y_j^{(i_1)} = y_j$ for $j \neq i_1$;

(ii) *for $l = 2, \dots, k$,*
$$y_{i_l}^{(i_1,\dots,i_l)}(x)\ y_{i_l}^{(i_1,\dots,i_{l-1})}(x+h) \ - \ y_{i_l}^{(i_1,\dots,i_l)}(x+h)\ y_{i_l}^{(i_1,\dots,i_{l-1})}(x) = T_{i_l}(x) \prod_{m=1,\ m\neq i_l}^{r} (y_m^{(i_1,\dots,i_{l-1})}(\,x+b_{i_l,m}\,))^{-\,a_{i_l,m}}$$

and $y_j^{(i_1,\dots,i_l)} = y_j^{(i_1,\dots,i_{l-1})}$ for $j \neq i_l$. □

The tuples $\boldsymbol{y}^{(i_1)}$, $\boldsymbol{y}^{(i_1,i_2)}$, $\dots$, $\boldsymbol{y}^{(i_1,\dots,i_k)}$ belong to the population $P_{\boldsymbol{y}}$. The tuple $\boldsymbol{y}^{(i_1)}$ is obtained from $\boldsymbol{y}$ by the i_1-th simple generation procedure and for $l = 2, \dots, k$ the tuple $\boldsymbol{y}^{(i_1,\dots,i_l)}$ is obtained from $\boldsymbol{y}^{(i_1,\dots,i_{l-1})}$ by the i_l-th simple generation procedure.

The sequence of tuples $\boldsymbol{y}^{(i_1)}$, $\boldsymbol{y}^{(i_1,i_2)}$, $\dots$, $\boldsymbol{y}^{(i_1,\dots,i_k)}$ satisfying Lemma 2.8 will be called *associated to the Bethe solution $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$.* The sequence of polynomials $y_{i_1}^{(i_1)}$, $y_{i_2}^{(i_1,i_2)}$, $\dots$, $y_{i_k}^{(i_1,\dots,i_k)}$ will be called *the diagonal sequence of polynomials associated to the Bethe solution $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$.* For a given $\boldsymbol{y}$ the diagonal sequence of polynomials determines the sequence of tuples $\boldsymbol{y}^{(i_1)}$, $\boldsymbol{y}^{(i_1,i_2)}$, $\dots$, $\boldsymbol{y}^{(i_1,\dots,i_k)}$ uniquely.

There are many diagonal sequences of polynomials associated to a given Bethe solution and a given sequence of indices.

3. Discrete Opers

In the remaining part of the paper, $\mathfrak{g} = \mathfrak{g}\,(A)$ is a simple Lie algebra of rank r.

Denote the coroots ${}^t\alpha_1^\vee, \dots, {}^t\alpha_r^\vee \in {}^t\mathfrak{h}$ of ${}^t\mathfrak{g}$ by $H_1, \dots, H_r$, respectively. Let $H_1, \dots, H_r$, $E_1, \dots, E_r$, $F_1, \dots, F_r$ be the Chevalley generators of ${}^t\mathfrak{g}$. We have $[H_j, E_i] = a_{i,j} E_i$ and $[H_j, F_i] = -a_{i,j} F_i$, where $A = (a_{i,j})$ is the Cartan matrix of $\mathfrak{g}$.

Let G be the complex simply connected Lie group with Lie algebra ${}^t\mathfrak{g}$. Let $B_\pm$, $N_\pm$ be the subgroups of G with Lie algebras ${}^t\mathfrak{b}_\pm$, ${}^t\mathfrak{n}_\pm$, respectively.

3.1. **Relations in G.** For a non-zero complex number u and $i \in \{1, \dots, r\}$, consider the elements u^{H_i}, $\exp(uE_i)$, $\exp(uF_i)$ in G. We will use the following relations.

Lemma 3.1. *Let u, v be non-zero complex numbers. Then*

$$u^{H_j}\ \exp(v\,E_i) \ = \ \exp(u^{a_{i,j}} v\,E_i)\ u^{H_j}\ ,$$

$$u^{H_j}\ \exp(v\,F_i) \ = \ \exp(u^{-a_{i,j}} v\,F_i)\ u^{H_j}\ ,$$

$$\exp(u\,F_i)\ \exp(v\,E_j) \ = \ \exp(v\,E_j)\ \exp(u\,F_i) \qquad \text{if}\ \ i \neq j\ ,$$

$$\exp(u\,F_i)\ \exp(v\,E_i) \ = \ \exp\Big(\frac{v}{1+uv}\,E_i\Big)\ (1+uv)^{-H_i}\ \exp\Big(\frac{u}{1+uv}\,F_i\Big)$$

if $1 + uv \neq 0$. □

3.2. **D-opers.** Define the shift operator ∂_h acting on functions of x by the formula

$$\partial_h\ :\ g(x)\ \mapsto\ g(x+h)\ .$$

A discrete oper (a d-oper) is a difference operator of the form $D = \partial - V$, where $V : \mathbb{C} \to G$ is a rational function.

For a rational function $s : \mathbb{C} \to N_+$, define the action of s on the d-oper by the formula

$$s \cdot D \ = \ s(x+h)\ D\ s(x)^{-1} \ = \ \partial_h \ - \ s(x+h)\ V(x)\ s(x)^{-1}\ .$$

The operator $s \cdot D$ is a d-oper. The d-opers D and $s \cdot D$ are called *gauge equivalent.*

3.3. **Miura d-opers associated to tuples of polynomials.** In the remaining part of the paper we assume that $\boldsymbol{b} = (b_{i,m})_{i,m=1,\ i\neq m}^r$ are special symmetric parameters given by (13).

Fix dominant integral weights $\boldsymbol{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ of $\mathfrak{g}$, complex numbers $\boldsymbol{z} = (z_1, \dots, z_n)$. Introduce polynomials $T_1(x), \dots, T_r(x)$ by formula (4).

For given polynomials $T_1, \dots, T_r$, a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ of non-zero polynomials, and $i \in \{1, \dots, r\}$, we define the rational function $R_{\boldsymbol{y},i}$ by the formula

$$R_{\boldsymbol{y},i}(x) \ = \ \frac{T_i(x)}{y_i(x+h)\ y_i(x)} \prod_{m,\ m\neq i} (y_m(\,x + b_{i,m}\,))^{-a_{i,m}}\ .$$

We say that a d-oper $D = \partial_h - V$ is *the Miura d-oper associated to weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$*, and the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ if

$$V(x) \ = \ \prod_{j=1}^r y_j(x+h)^{-H_j}$$

$$\times\ \ \exp(R_{\boldsymbol{y},1}(x)\,F_1)\exp(R_{\boldsymbol{y},2}(x)\,F_2)\cdots\exp(R_{\boldsymbol{y},r}(x)\,F_r)\prod_{j=1}^r y_j(x)^{H_j}\ .$$

We denote by $D_{\boldsymbol{y}} = \partial_h - V_{\boldsymbol{y}}$ the Miura d-oper associated to the tuple $\boldsymbol{y}$.

It is easy to see that if a Miura d-oper D is associated to weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ of non-zero polynomials, then the tuple $\boldsymbol{y}$ is determined uniquely by D.

It follows easily from Lemma 3.1 that the d-oper $D_{\boldsymbol{y}}$ does not change if the polynomials of the tuple $\boldsymbol{y}$ are multiplied by non-zero numbers.

For $i \in \{1, \dots, r\}$, we say that the Miura d-oper $D_{\boldsymbol{y}}$ *is deformable in the i-th direction* if there exists a non-zero rational function $g_i : \mathbb{C} \to \mathbb{C}$ and a non-zero polynomial $\tilde{y}_i$ such that

$$\exp(g_i(x+h)\, E_i)\ D_{\boldsymbol{y}}\ \exp(-g_i(x)\, E_i)\ =\ D_{\boldsymbol{y}^{(i)}}\ ,$$

where $\boldsymbol{y}^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$. We say that $D_{\boldsymbol{y}}$ *is deformed to $D_{\boldsymbol{y}^{(i)}}$ with the help of g_i.*

Theorem 3.1. *Let $\boldsymbol{b}$ be the special symmetric parameters. Let $\boldsymbol{y}$ be a tuple of non-zero polynomials.*

(i) *Assume that equation (7) has a polynomial solution $\tilde{y}_i$. Set $\boldsymbol{y}^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$. Then the Miura d-oper $D_{\boldsymbol{y}}$ is deformable in the i-th direction to the Miura d-oper $D_{\boldsymbol{y}^{(i)}}$ with the help of*

$$g_i(x)\ =\ \frac{1}{y_i(x)\, \tilde{y}_i(x)} \prod_{m,\ m\neq i} y_m(x)^{-a_{i,m}}\ . \tag{14}$$

(ii) *If the tuple $\boldsymbol{y}$ is generic in the sense of Section 2.3 and the Miura d-oper $D_{\boldsymbol{y}}$ is deformable in the i-th direction to the Miura d-oper $D_{\boldsymbol{y}^{(i)}}$ with the help of g_i, then $\tilde{y}_i$ is a polynomial solution of equation (7), and $g_i, \tilde{y}_i$ satisfy (14).*

Proof. For a scalar rational function g_i we have

$$\begin{aligned} \exp(g_i(x+h)\, E_i)\ D_{\boldsymbol{y}}\ \exp(-g_i(x)\, E_i)\ &=\ \partial_h - \\ \prod_{j=1}^{r} y_j(x+h)^{-H_j} \exp\left(R_{\boldsymbol{y},1}(x)\, F_1\right)\ &\exp\left(R_{\boldsymbol{y},2}(x)\, F_2\right)\ \cdots \\ \exp(\tilde{g}_i(x+h)\, E_i)\ &\exp\left(R_{\boldsymbol{y},i}(x)\, F_i\right)\ \exp(-\tilde{g}_i(x)\, E_i)\ \cdots \\ &\exp\left(R_{\boldsymbol{y},r}(x)\, F_r\right)\ \prod_{j=1}^{r} y_j(x)^{H_j}\ , \end{aligned}$$

where

$$\tilde{g}_i(x)\ =\ g_i(x) \prod_{m=1}^{r} y_m(x)^{a_{i,m}}\ .$$

By Lemma 3.1 the Miura d-oper D is deformable in the i-th direction only if

$$\tilde{g}_i(x+h) = \tilde{g}_i(x)/(1 - \tilde{g}_i(x) R_{\boldsymbol{y},i}(x)).$$

This equation is called *the i-th discrete Ricatti equation*, see [MV4] where the classical Ricatti equation appears in an analogous situation.

The Ricatti equation can be written as

$$y_i(x+h)\,\frac{y_i(x)}{\tilde g_i(x)}\;-\;y_i(x)\,\frac{y_i(x+h)}{\tilde g_i(x+h)}\;=\;T_i(x)\prod_{m,\ m\neq i} y_m(x+b_{i,m})^{-a_{i,m}}\ . \tag{15}$$

If equation (7) has a polynomial solution $\tilde y_i$, then (15) has a rational solution

$$\tilde g_i(x)\;=\;\frac{y_i(x)}{\tilde y_i(x)}.$$

Then g_i is given by (14), and

$$1\;-\;\tilde g_i(x)\;R_{\boldsymbol{y},i}(x)\;=\;\frac{\tilde y_i(x+h)}{y_i(x+h)}\,\frac{y_i(x)}{\tilde y_i(x)}, \tag{16}$$

$$\begin{aligned}
&\exp(\tilde g_i(x+h)\,E_i)\ \exp\left(R_{\boldsymbol{y},i}(x)\,F_i\right)\ \exp(-\tilde g_i(x)\,E_i)\;=\\
&\left(\frac{y_i(x+h)}{\tilde y_i(x+h)}\right)^{H_i}\left(\frac{\tilde y_i(x)}{y_i(x)}\right)^{H_i}\times\\
&\exp\left(\frac{\tilde y_i(x)\ T_i(x)}{\tilde y_i(x+h)\ (y_i(x))^2}\prod_{m,\ m\neq i} y_m(x+b_{i,m})^{-a_{i,m}}\,F_i\right)=\\
&\left(\frac{y_i(x+h)}{\tilde y_i(x+h)}\right)^{H_i}\exp\left(R_{\boldsymbol{y}^{(i)},i}(x)\,F_i\right)\ \left(\frac{\tilde y_i(x)}{y_i(x)}\right)^{H_i}.
\end{aligned} \tag{17}$$

Using the last formula and Lemma 3.1 we easily conclude that the Miura d-oper $D_{\boldsymbol{y}}$ is deformed in the i-th direction to the Miura d-oper $D_{\boldsymbol{y}^{(i)}}$ with the help of g_i given by (14) if $\tilde y_i$ is a polynomial solution of (7). This proves part (i) of the theorem.

To prove part (ii) write (15) as

$$\frac{1}{\tilde g_i(x)}\;-\;\frac{1}{\tilde g_i(x+h)}\;=\frac{T_i(x)}{y_i(x+h)\ y_i(x)}\prod_{m,\ m\neq i} y_m(x+b_{i,m})^{-a_{i,m}}\ .$$

Let $\tilde g_i(x)$ be a rational solution of this equation. Since $\boldsymbol{y}$ is generic, the poles of $1/\tilde g_i(x)$ are located at zeros of $y_i(x)$ and all poles are simple. Hence $\tilde y_i(x) = y_i(x)/\tilde g_i(x)$ is a polynomials solution of (7). Then formulas (16), (17) hold and part (ii) is proved. □

Corollary 3.2. *Let the Miura d-oper $D_{\boldsymbol{y}}$ be associated to weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1,\dots,y_r)$. Assume that the tuple $\boldsymbol{y} = (y_1,\dots,y_r)$ is generic in the sense of Section 2.3. Then $D_{\boldsymbol{y}}$ is deformable in all directions from 1 to r if and only if the tuple $\boldsymbol{y}$ represents a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z},\ \boldsymbol{\Lambda},\ \Lambda_\infty = \sum_{i=1}^n \Lambda_i - \sum_{i=1}^r l_i\alpha_i,\ l_i = \deg y_i,$ and special symmetric parameters $\boldsymbol{b}$.*

Let the Miura d-oper $D_{\boldsymbol{y}}$ be associated to weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$. Let the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ represent a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. Let ${\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$ be the variety of all Miura d-opers each of which can be obtained from $D_{\boldsymbol{y}}$ by a sequence of deformations in directions $i_1, \dots, i_k$ where k is a positive integer and all i_j lie in $\{1, \dots, r\}$.

Corollary 3.3. *The variety ${\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$ is isomorphic to the population $P_{\boldsymbol{y}}$ of solutions of Bethe ansatz equations, where $P_{\boldsymbol{y}}$ is the population originated at $\boldsymbol{y}$.*

4. Solutions of Difference Equations

Let $D_{\boldsymbol{y}} = \partial_h - V_{\boldsymbol{y}}$ be the Miura d-oper associated to a Bethe solution $\boldsymbol{y}$ of the Bethe ansatz equations associated to special symmetric parameters $\boldsymbol{b}$. Let $P_{\boldsymbol{y}}$ be the population of solutions originated at $\boldsymbol{y}$.

In this section we prove that the difference equation

$$Y(x+h) \ = \ V_{\boldsymbol{y}}(x)\ Y(x) \tag{18}$$

has a G-valued rational solution. We will write that solution explicitly in terms of coordinates of tuples composing the population.

Note that if $Y(x)$ is a solution and $g \in G$, then $Y(x)g$ is a solution too.

First we give a formula for a solution of equation (18) for d-opers associated to $\mathfrak{g}$ of type A_r, and then we consider more general formulas for solutions which do not use the structure of the Lie algebra.

Let Y be a solution of equation (18). Define

$$\bar{Y}(x) \ = \ \prod_{j=1}^{r} y_j(x)^{H_j}\ Y(x)\ .$$

Then $\bar{Y}$ is a solution of the equation

$$\bar{Y}(x+h) \ = \ \bar{V}_{\boldsymbol{y}}(x)\ \bar{Y}(x) \tag{19}$$

where

$$\bar{V}_{\boldsymbol{y}}(x) = \exp\left(R_{\boldsymbol{y},1}(x)\, F_1\right)\, \exp\left(R_{\boldsymbol{y},2}(x)\, F_2\right) \cdots \exp\left(R_{\boldsymbol{y},r}(x)\, F_r\right).$$

4.1. **The A_r d-opers and solutions of Bethe ansatz equations.** In this section let $\mathfrak{g} = sl_{r+1}$ be the Lie algebra of type A_r. Then ${}^t\mathfrak{g} = sl_{r+1}$. We have $(\alpha_i, \alpha_i) = 2$ for all i. We fix the order of simple roots of sl_{r+1} such that

$$(\alpha_1, \alpha_2) = (\alpha_2, \alpha_3) = \cdots = (\alpha_{r-1}, \alpha_r) = -1.$$

We start with two examples.

Let $\mathfrak{g} = sl_2$. Let $\boldsymbol{y} = (y_1)$ represent a Bethe solution of the sl_2 Bethe ansatz equations associated to $\boldsymbol{z}$, $\boldsymbol{\Lambda}$, Λ_∞. Let $y_1^{(1)}$ be the diagonal sequence of polynomials associated to $\boldsymbol{y}$ and the sequence of indices (1), in other words,

$$y_1(x+h)\ y_1^{(1)}(x)\ -\ y_1(x)\ y_1^{(1)}(x+h)\ =\ T_1(x)\ .$$

Then

$$\bar{Y}\ =\ \exp\,(\frac{y_1^{(1)}}{y_1}F_1)$$

is a solution of the difference equation (19) with values in $\mathrm{SL}\,(2,\mathbb{C})$. Indeed,

$$\begin{aligned}
&(\partial_h - \exp(\frac{T_1(x)}{y_1(x)y_1(x+h)}\,F_1))\ \exp\,(\frac{y_1^{(1)}(x)}{y_1(x)}F_1)\ =\ \exp\,(\frac{y_1^{(1)}(x+h)}{y_1(x+h)}F_1)\times \\
&(\partial_h - \exp(\,(-\frac{y_1^{(1)}(x+h)}{y_1(x+h)} + \frac{y_1^{(1)}(x)}{y_1(x)} + \frac{T_1(x)}{y_1(x)y_1(x+h)})\,F_1\,)\,)\ = \\
&\exp\,(\frac{y_1^{(1)}(x+h)}{y_1(x+h)}F_1)\ (\partial_h - \mathrm{id})\ .
\end{aligned}$$

Let $\mathfrak{g} = sl_3$. Let $\boldsymbol{y} = (y_1, y_2)$ represent a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. Let $y_1^{(1)}, y_2^{(1,2)}$ be the diagonal sequence of polynomials associated to $\boldsymbol{y}$ and the sequence of indices $(1,2)$, in other words,

$$\begin{aligned}
y_1(x+h)\ y_1^{(1)}(x)\ -\ y_1(x)\ y_1^{(1)}(x+h)\ &=\ T_1(x)\ y_2(x+h)\ , \\
y_2(x+h)\ y_2^{(1,2)}(x)\ -\ y_2(x)\ y_2^{(1,2)}(x+h)\ &=\ T_2(x)\ y_1^{(1)}(x)\ .
\end{aligned}$$

Let $y_2^{(2)}$ be the diagonal sequence of polynomials associated to $\boldsymbol{y}$ and the sequence of indices (2), in other words,

$$y_2(x+h)\ y_2^{(2)}(x)\ -\ y_2(x)\ y_2^{(2)}(x+h)\ =\ T_2(x)\ y_1(x)\ .$$

Then

$$\bar{Y}(x)\ =\ \exp\,(\frac{y_1^{(1)}(x)}{y_1(x)}F_1)\ \exp\,(\frac{y_2^{(1,2)}(x)}{y_2(x)}[F_2,F_1])\ \exp\,(\frac{y_2^{(2)}(x)}{y_2(x)}F_2)$$

is a solution of the difference equation (19) with values in $\mathrm{SL}\,(3,\mathbb{C})$. Indeed, we have

$$(\partial_h - \bar V_{\boldsymbol{y}}(x))\ \exp\,(\frac{y_1^{(1)}(x)}{y_1(x)}F_1)\ =$$

$$\exp(\frac{y_1^{(1)}(x+h)}{y_1(x+h)}F_1)(\partial_h - \exp((-\frac{y_1^{(1)}(x+h)}{y_1(x+h)} + \frac{y_1^{(1)}(x)}{y_1(x)} + \frac{T_1(x)y_2(x+h)}{y_1(x)y_1(x+h)})F_1)\times$$

$$\exp\,(\frac{T_2(x)y_1^{(1)}(x)}{y_2(x)y_2(x+h)}[F_2,F_1])\ \exp\,(\frac{T_2(x)y_1(x)}{y_2(x)y_2(x+h)}F_2)\ ) =$$

$$\exp(\frac{y_1^{(1)}(x+h)}{y_1(x+h)}F_1)\times$$

$$(\partial_h - \exp(\frac{T_2(x)y_1^{(1)}(x)}{y_2(x)y_2(x+h)}[F_2,F_1])\exp(\frac{T_2(x)y_1(x)}{y_2(x)y_2(x+h)}F_2)),$$

$$(\partial_h - \exp(\frac{T_2(x)y_1^{(1)}(x)}{y_2(x)y_2(x+h)}[F_2,F_1])\exp(\frac{T_2(x)y_1(x)}{y_2(x)y_2(x+h)}F_2))\times$$

$$\exp\,(\frac{y_2^{(1,2)}(x)}{y_2(x)}[F_2,F_1]) =$$

$$\exp\,(\frac{y_2^{(1,2)}(x+h)}{y_2(x+h)}[F_2,F_1])\ (\partial_h - \exp(\frac{T_2(x)y_1(x)}{y_2(x)y_2(x+h)}F_2)),$$

and

$$(\partial_h - \exp(\frac{T_2(x)y_1(x)}{y_2(x)y_2(x+h)}F_2))\ \exp\,(\frac{y_2^{(2)}(x)}{y_2(x)}F_2) =$$

$$\exp\,(\frac{y_2^{(2)}(x+h)}{y_2(x+h)}F_2)\ (\partial_h - \mathrm{id})\ .$$

Consider the general case. Let $\mathfrak{g} = sl_{r+1}$. Let $\boldsymbol{y} = (y_1,\dots,y_r)$ represent a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z},\boldsymbol{\Lambda},\Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. For $i=1,\dots,r$, let $y_i^{(i)},\ y_{i+1}^{(i,i+1)},\ \dots,\ y_r^{(i,\dots,r)}$ be the diagonal sequence of polynomials associated to $\boldsymbol{y}$ and the sequence of indices $(i,i+1,\dots,r)$, in other words,

$$y_i(x+h)\ y_i^{(i)}(x)\ -\ y_i(x)\ y_i^{(i)}(x+h)\ =\ T_i(x)\ y_{i-1}(x)\ y_{i+1}(x+h)\ ,$$

$$y_{i+1}(x+h)y_{i+1}^{(i,i+1)}(x) - y_{i+1}(x)y_{i+1}^{(i,i+1)}(x+h) = T_{i+1}(x)\ y_i^{(i)}(x)y_{i+2}(x+h),\quad \dots,$$

$$y_{r-1}(x+h)y_{r-1}^{(i,\dots,r-1)}(x) - y_{r-1}(x)y_{r-1}^{(i,\dots,r-1)}(x+h) = T_{r-1}(x)\ y_{r-2}^{(i,\dots,r-2)}(x)y_r(x+h),$$

$$y_r(x+h)\ y_r^{(i,\dots,r)}(x) - y_r(x)\ y_r^{(i,\dots,r)}(x+h)\ =\ T_r(x)\ y_{r-1}^{(i,\dots,r-1)}(x).$$

Define r functions $Y_1,\dots,Y_r$ of x with values in $\mathrm{SL}\,(r+1,\mathbb{C})$ by the formulas

$$Y_i(x) \;=\; \prod_{j=i}^{r} \exp\,(\,\frac{y_j^{(i,\dots,j)}(x)}{y_j(x)}\;[F_j,[F_{j-1},[...,[F_{i+1},F_i]...]]]\,)\;.$$

Note that inside each product the factors commute.

Theorem 4.1. *The product $Y_1\cdots Y_r$ is a solution of the difference equation (19) with values in $\mathrm{SL}\,(r+1,\mathbb{C})$.*

The proof is straightforward. One shows that

$$\begin{aligned}(\partial_h \;-\; \exp\,(R_{\boldsymbol{y},i}(x)\,F_i)\cdots\exp\,(R_{\boldsymbol{y},r}(x)\,F_r))\;Y_i(x) \;=&\\ Y_i(x+h)\;(\partial_h \;-\; \exp\,(R_{\boldsymbol{y},i+1}(x)\,F_{i+1})\cdots\exp\,(R_{\boldsymbol{y},r}(x)\,F_r))\;.&\end{aligned}$$

4.2. **General formulas for solutions.** Let U be a complex finite dimensional representation of G. Let u_{low} be a lowest weight vector of U, $\;{}^t\mathfrak{n}_-\,u_{\mathrm{low}}=0$.

Let $\boldsymbol{y}=(y_1,\dots,y_r)$ represent a Bethe solutions of the Bethe ansatz equations associated to $\boldsymbol{z},\boldsymbol{\Lambda},\Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. We solve the difference equation (18) with values in U.

Let $\boldsymbol{i}=(i_1,i_2,\dots,i_k)$, $1\le i_j\le r$, be a sequence of natural numbers. Let $\boldsymbol{y}^{(i_1)}=(y_1^{(i_1)},\dots,y_r^{(i_1)})$, $\boldsymbol{y}^{(i_1,i_2)}=(y_1^{(i_1,i_2)},\dots,y_r^{(i_1,i_2)})$, $\dots$, $\boldsymbol{y}^{(i_1,\dots,i_k)}=(y_1^{(i_1,\dots,i_k)},\dots,y_r^{(i_1,\dots,i_k)})$ be a sequence of tuples associated to the Bethe solution $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$.

Theorem 4.2. *The U-valued function*

$$\begin{aligned} Y(x) \;&=\; \exp\left(-\frac{1}{y_{i_1}(x)\,y_{i_1}^{(i_1)}(x)}\prod_{m,\,m\neq i_1}(y_m(x))^{-a_{i_1,m}}E_{i_1}\right)\\ &\times\;\exp\left(-\frac{1}{y_{i_2}^{(i_1)}(x)\,y_{i_2}^{(i_1,i_2)}(x)}\prod_{m,\,m\neq i_2}(y_m^{(i_1)}(x))^{-a_{i_2,m}}E_{i_2}\right)\;\cdots\\ &\times\;\exp\left(-\frac{1}{y_{i_k}^{(i_1,\dots,i_{k-1})}(x)y_{i_k}^{(i_1,\dots,i_k)}(x)}\prod_{m,\,m\neq i_k}(y_m^{(i_1,\dots,i_{k-1})}(x))^{-a_{i_k,m}}E_{i_k}\right)\\ &\times\;\prod_{j=1}^{r}(y_j^{(i_1,\dots,i_k)}(x))^{-H_j}\;\;u_{\mathrm{low}}\;.\end{aligned}$$

is a solution of the difference equation (18).

The proof is straightforward and follows from Theorem 3.1.

Corollary 4.1. *Every coordinate of every solution of the difference equation (18) with values in a finite dimensional representation of G can be written as a rational function $R(f_1,\dots,f_N)$ of suitable polynomials $f_1,\dots,f_N$ which appear as coordinates of tuples in the $\mathfrak{g}$ population $P_{\boldsymbol{y}}$ generated at $\boldsymbol{y}$.*

Since G has a faithful finite dimensional representation, the solutions of the difference equation (18) with values in G also can be written as rational functions of coordinates of tuples of $P_{\boldsymbol{y}}$, cf. Section 4.1.

Corollary 4.2. *Let $\boldsymbol{y} = (y_1, \dots, y_r)$ represent a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. Then there exists a G-valued rational function $Y : \mathbb{C} \to G$ satisfying equation (18).*

5. Miura d-opers and flag varieties

5.1. **Theorem on isomorphism.** Let $\boldsymbol{y} = (y_1, \dots, y_r)$ represent a Bethe solution of the Bethe ansatz equations associated to $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$, and special symmetric parameters $\boldsymbol{b}$. Let $D_{\boldsymbol{y}} = \partial_h - V_{\boldsymbol{y}}$ be the Miura d-oper associated to $\boldsymbol{y}$. Consider the variety $\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}$ of all Miura d-opers gauge equivalent to $D_{\boldsymbol{y}}$. If $D' \in \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}$, then there exists a rational function $v : \mathbb{C} \to N_+$ such that $D' \;=\; v(x+h)\, D_{\boldsymbol{y}}\, v(x)^{-1}$. In that case we denote D' by D^v.

The variety of pairs

$$\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}} \;=\; \{(D^v, v) \mid D^v \in \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}\}$$

will be called *the variety of marked Miura d-opers gauge equivalent to* $D_{\boldsymbol{y}}$. We have the natural projection $\pi : \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}} \to \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}},\ (D^v, v) \mapsto D^v$. We will show below that π is an isomorphism.

Let ${\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0 \subseteq \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}$ be the subvariety of all Miura d-opers each of which can be obtained from $D_{\boldsymbol{y}}$ by a sequence of deformations in directions $i_1, \dots, i_k$ where k is a non-negative integer and all i_j lie in $\{1, \dots, r\}$. By Corollary 3.3 the subvariety ${\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$ is isomorphic to the population of solutions of Bethe ansatz equations associated to special symmetric parameters $\boldsymbol{b}$ and originated at $\boldsymbol{y}$. We will show below that ${\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0 = \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}$.

Assume that $D' \in {\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$ and D' is obtained from $D_{\boldsymbol{y}}$ by a sequence of deformations in directions $i_1, \dots, i_k$ where k is a non-negative integer and all i_j lie in $\{1, \dots, r\}$. Then there exist scalar rational functions $g_1, \dots, g_k$ with the following properties. For $j = 1, \dots, k$, define a rational N_+-valued function $v_j : \mathbb{C} \to N_+$,

$$v_j(x) \;=\; \exp\left(g_j(x) E_{i_j}\right) \cdots \exp\left(g_2(x) E_{i_2}\right) \exp\left(g_1(x) E_{i_1}\right) \,. \tag{20}$$

Then $D^{v_j} \in {\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$ and $D' = D^{v_k}$. The set of all pairs (D^{v_k}, v_k) such that k is a non-negative integer, v_k is given by the above construction, and $D^{v_k} \in {\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$, will be called *the variety of specially marked Miura d-opers gauge equivalent to* $D_{\boldsymbol{y}}$ and denoted by ${\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}}^0$. Clearly we have ${\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}}^0 \subseteq \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}$.

Let $\mathbb{P}^1$ be the complex projective line. Consider $D_{\boldsymbol{y}}$ as a discrete connection $\nabla_{\boldsymbol{y}}$ on the trivial principal G-bundle $p : G \times \mathbb{P}^1 \to \mathbb{P}^1$. Namely, by definition a section

$$U \;\to\; G \times U, \qquad x \;\mapsto\; Y(x) \times x,$$

of p over a subset $U \subset \mathbb{C} \subset \mathbb{P}^1$ is called *horizontal* if the G-valued function $Y(x)$ is a solution of the difference equation (18), $Y(x+h) = V_{\boldsymbol{y}}(x)\,Y(x)$. By Corollary 4.2 equation (18) has a rational solution $Y(x)$. For any $g \in G$ the rational G-valued function $Y(x)g$ is a solution of the same equation too. A point $x_0 \in \mathbb{C}$ will be called *regular* if x_0 is a regular point of the rational functions $Y(x)$ and $V_{\boldsymbol{y}}(x)$.

Let $x_0 \in \mathbb{C}$ be a regular point. Let g be an element of G. Then $\nabla_{\boldsymbol{y}}$ has a rational horizontal section s such that $s(x_0) = g$.

It is easy to see that if the values of two rational horizontal sections are equal at one point, then the sections are equal.

Consider the trivial bundle $p' : (G/B_-) \times \mathbb{P}^1 \to \mathbb{P}^1$ associated to the bundle p. The fiber of p' is the flag variety G/B_-. The discrete connection $\nabla_{\boldsymbol{y}}$ induces a discrete connection $\nabla'_{\boldsymbol{y}}$ on p'.

The variety Γ of rational horizontal sections of the discrete connection $\nabla'_{\boldsymbol{y}}$ is identified with the fiber $(p')^{-1}(x_0)$ over any regular point x_0. Thus, Γ is isomorphic to G/B_-.

Any G-valued rational function v defines a section

$$S_v \ : \ x \ \mapsto \ v(x)^{-1}B_- \times x \tag{21}$$

of p' over the set of regular points of v. The section S_v is also well defined over the poles of v since G/B_- is a projective variety.

If $D^v \in \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}$, then the section S_v is horizontal with respect to $\nabla'_{\boldsymbol{y}}$. This follows from the fact that the function $V_{\boldsymbol{y}}$ takes values in B_-. Thus we have a map

$$S \ : \ \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}} \to \Gamma, \qquad (D^v, v) \mapsto S_v \ .$$

Theorem 5.1. *The map $S : \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}} \to \Gamma$ is an isomorphism and $\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^{\,0} = \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}$.*

Proof. Let $(D^{v_1}, v_1), (D^{v_2}, v_2) \in \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}$. Assume that the images of (D^{v_1}, v_1) and (D^{v_2}, v_2) under the map S coincide. Assume that $v_1, v_2, V_{\boldsymbol{y}}$ are regular at $x_0 \in \mathbb{C}$. The equality $S_{v_1}(x_0) = S_{v_2}(x_0)$ means that $v_1(x_0)^{-1}B_- = v_2(x_0)^{-1}B_-$. Then $v_1(x_0) = v_2(x_0)$. Hence $v_1 = v_2$ and $D^{v_1} = D^{v_2}$. That proves the injectivity of S.

Let x_0 be a regular point of $V_{\boldsymbol{y}}$ in $\mathbb{C}$. For any $u \in N_+$ there exists a rational N_+-valued function v such that $v(x_0) = u$, $D^v \in {\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^0$. Indeed, every $u \in N_+$ is a product of elements of the form $e^{c_i E_i}$ for $i \in \{1, \dots, r\}$ and $c_i \in \mathbb{C}$. Every c_i can be taken as the initial condition for a solution of the suitable i-th discrete Ricatti equation.

Thus the set

$$Im(x_0) \ = \ \{S_v(x_0) \in \ (G/B_-) \times x_0 \mid (D^v, v) \in \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^{\,0}\}$$

contains the set $((N_+B_-)/B_-) \times x_0 \ \subset \ (G/B_-) \times x_0$. It is easy to see that the set $Im(x_0)$ is closed in $(G/B_-) \times x_0$ as the image of $\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^{\,0}$ with respect to S. On the other hand the set $((N_+B_-)/B_-) \times x_0$ is dense in $(G/B_-) \times x_0$. Hence $Im(x_0) = (G/B_-) \times x_0$, and $\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}^{\,0} = \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}}}}$ since the map S is injective. □

5.2. **Remarks on the isomorphism.** Let $\mathfrak{g}$ be a simple Lie algebra. Let $\boldsymbol{y}^0$ be a Bethe solution of the Bethe ansatz equations associated to special symmetric parameters $\boldsymbol{b}$. Theorem 5.1 says that the variety $\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}$ is isomorphic to the flag variety G/B_-. Here are some comments on that isomorphism.

The isomorphism is constructed in two steps. If $(D^v, v) \in \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}$ is a marked Miura d-oper, then we assign to it the section $S_v \in \Gamma$ by formula (21). We choose a regular point $x_0 \in \mathbb{C}$, and assign to the section S_v its value $S_v(x_0) \in (G/B_-) \times x_0$ at x_0. The resulting composition

$$\phi_{\boldsymbol{y}^0, x_0} \ : \ \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}} \ \to \ G/B_-$$

is an isomorphism according to Theorem 5.1.

Lemma 5.1. *If $x_0, x_1 \in \mathbb{C}$ are regular points, then there exists an element $g \in B_-$ such that $\phi_{\boldsymbol{y}^0, x_1} \ = \ g \ \phi_{\boldsymbol{y}^0, x_0}$.*

Proof. Let Y be the G-valued rational solution of equation (18) such that $Y(x_0) = \mathrm{id}$. Then $Y(x) \in B_-$ for all x. If $(D^v, v) \in \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}$, then S_v is a horizontal section of $\nabla'_{\boldsymbol{y}^0}$. Thus it has the form $x \ \mapsto \ (Y(x)uB_-) \times x$ for a suitable element $u \in G$. Hence $\phi_{\boldsymbol{y}^0, x_0}(\boldsymbol{y}) \ = \ Y(x_0)uB_-$ and $\phi_{\boldsymbol{y}^0, x_1}(\boldsymbol{y}) \ = \ Y(x_1)uB_-$. We conclude that $\phi_{\boldsymbol{y}^0, x_1} \ = \ Y(x_1)Y(x_0)^{-1} \, \phi_{\boldsymbol{y}^0, x_0}$. □

6. Bruhat Cells

6.1. **Properties of Bruhat cells.** Let $\mathfrak{g}$ be a simple Lie algebra. For an element w of the Weyl group W, the set

$$B_w \ = \ B_- w B_- \ \subset \ G/B_-$$

is called *the Bruhat cell* associated to w. The Bruhat cells form a cell decomposition of the flag variety G/B_-.

For $w \in W$ denote $l(w)$ the length of w. We have $\dim B_w = l(w)$.

Let $s_1, \dots, s_r \in W$ be the generating reflections of the Weyl group. For $v \in G/B_-$ and $i \in \{1, \dots, r\}$ consider the rational curve

$$\mathbb{C} \ \to \ G/B_-, \qquad c \ \mapsto \ e^{cE_i} v \ .$$

The limit of $e^{cE_i}\, v$ is well defined as $c \to \infty$, since G/B_- is a projective variety.

We need the following standard property of Bruhat cells.

Lemma 6.1. *Let $s_i, w \in W$ be such that $l(s_i w) = l(w) + 1$. Then*

$$B_{s_i w} \ = \ \{ \ e^{cE_i} v \ | \ v \in B_w, \ c \in \{\mathbb{P}^1 - 0\} \ \} \ .$$

□

Corollary 6.2. *Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of $w \in W$. Then*

$$B_w = \{ \lim_{c_1 \to c_1^0} \dots \lim_{c_k \to c_k^0} e^{c_1 E_{i_1}} \cdots e^{c_k E_{i_k}} B_- \ \in \ G/B_- \ | \ c_1^0, \dots, c_k^0 \in \{\mathbb{P}^1 - 0\} \}.$$

Corollary 6.3. *Let $s_{i_1}\cdots s_{i_k}$ be an element in W. Let $c_1^0,\dots,c_k^0\in\mathbb{P}^1$. Then the element*

$$\lim_{c_1\to c_1^0}\dots\lim_{c_k\to c_k^0} e^{c_1E_{i_1}}\cdots e^{c_kE_{i_k}}B_-\ \in\ G/B_-$$

belongs to the union of the Bruhat cells B_w with $l(w)\le k$.

6.2. **Populations and Bruhat cells.** Let P be a population of solutions of the Bethe ansatz equations associated to integral dominant weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, special symmetric parameters $\boldsymbol{b}$.

Let $\boldsymbol{y}^0=(y_1^0,\dots,y_r^0)\in P$ be a point of the population with $l_i=\deg\, y_i^0$ for $i=1,\dots,r$. Assume that the weight at infinity of $\boldsymbol{y}^0$,

$$\Lambda_\infty\ =\ \sum_{i=1}^n\Lambda_i\ -\ \sum_{i=1}^r\ l_i\,\alpha_i\ ,$$

is integral dominant, see Section 2. Such $\boldsymbol{y}^0$ exists according to [MV1], [MV2]. For $w\in W$ consider the weight $w\cdot\Lambda_\infty$, where $w\cdot$ is the shifted action of w on $\mathfrak{h}^*$. Write

$$w\cdot\Lambda_\infty\ =\ \sum_{i=1}^n\Lambda_i\ -\ \sum_{i=1}^r\ l_i^w\,\alpha_i.$$

Set

$$P_w\ =\ \{\ \boldsymbol{y}=(y_1,\dots,y_r)\ \in\ P\ |\ \deg y_i=l_i^w,\ i=1,\dots,r\ \}\ .$$

Clearly, $P=\cup_{w\in W}P_w$, and $P_{w_1}\cap P_{w_2}=\emptyset$ if $w_1\ne w_2$.

Consider the trivial bundle $p':(G/B_-)\times\mathbb{P}^1\to\mathbb{P}^1$ with the discrete connection $\nabla'_{\boldsymbol{y}^0}$. Consider the Bruhat cell decomposition of fibers of p'.

Assume that $x_0\in\mathbb{C}$ is a regular point of the Miura d-oper $D_{\boldsymbol{y}^0}=\partial_h-V_{\boldsymbol{y}^0}$ and x_0 is a regular point of the G-valued rational solutions of the associated difference equation (18), $Y(x+h)\ =\ V_{\boldsymbol{y}^0}(x)\,Y(x)$. Let

$$\phi_{\boldsymbol{y}^0,x_0}\ :\ \widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}\ \to\ G/B_-$$

be the isomorphism defined in Section 5.2. Let

$$\pi:\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}\to\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}},\qquad (D^v,v)\mapsto D^v,$$

be the natural projection. Let

$$\xi:\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}\to P,\qquad D_{\boldsymbol{y}}\mapsto\boldsymbol{y},$$

be the isomorphism of Corollary 3.3.

Theorem 6.1. *For every $w\in W$, the composition $\xi\pi\phi^{-1}_{\boldsymbol{y}^0,x_0}:G/B_-\to P$, restricted to the Bruhat cell $B_{w^{-1}}$, is a 1-1 epimorphism of $B_{w^{-1}}$ onto P_w.*

Corollary 6.4. *The projection $\pi:\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}\to\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}$ is an isomorphism, i.e. if (D^{v_1},v_1), $(D^{v_2},v_2)\in\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}$ are such that $D^{v_1}=D^{v_2}$, then $v_1=v_2$.*

Corollary 6.5. *Let P be a population of solutions of the Bethe ansatz equations associated to integral dominant $\mathfrak{g}$-weights $\Lambda_1, \dots, \Lambda_n$, complex numbers $z_1, \dots, z_n$, special symmetric parameters $\boldsymbol{b}$. Then P is isomorphic to the flag variety G/B_- of the Langlands dual algebra ${}^t\mathfrak{g}$.*

Corollary 6.6. *Let $\Lambda_1, \dots, \Lambda_n, \Lambda_\infty$ be integral dominant $\mathfrak{g}$-weights. Let $z_1, \dots, z_n$ be complex numbers. Let $w \in W$. Consider the Bethe ansatz equations associated to $\Lambda_1, \dots, \Lambda_n, w \cdot \Lambda_\infty = \sum_{i=1}^n \Lambda_i - \sum_{i=1}^r l_i^w \alpha_i$, $z_1, \dots, z_n$, and special symmetric parameters $\boldsymbol{b}$. A solution of the Bethe ansatz equations is a collection of complex numbers $\boldsymbol{t} = (t_j^{(i)})$, $i = 1, \dots, r$, $j = 1, \dots, l_i^w$. Let K be a connected component of the set of solutions of the Bethe ansatz equations. For each $\boldsymbol{t} \in K$ consider the tuple $\boldsymbol{y_t} \in (\mathbb{C}[x])^r$ of monic polynomials representing the solution $\boldsymbol{t}$. Then the closure of the set $\{ \boldsymbol{y_t} \mid \boldsymbol{t} \in K \}$ in $(\mathbb{C}[x])^r$ is an $l(w)$-dimensional cell.*

6.3. **Proof of Theorem 6.1.**

Lemma 6.7. *For $w \in W$, the subset $B_w \times \mathbb{P}^1 \subset (G/B_-) \times \mathbb{P}^1$ is invariant with respect to the discrete connection $\nabla'_{\boldsymbol{y}^0}$.*

Proof. Let Y be the rational G-valued solution of the equation $D_{\boldsymbol{y}^0} Y = 0$ such that $Y(x_0) = \mathrm{id}$. Then $Y(x) \in B_-$ for all x. The rational horizontal sections of $\nabla'_{\boldsymbol{y}^0}$ have the form $x \mapsto (Y(x)\, uB_-) \times x$ for a suitable element $u \in G$. If $uB_- \in B_w$, then $Y(x) uB_- \in B_w$ for all x. $\square$

Let $w = s_{i_k} \cdots s_{i_1}$ be a reduced decomposition of $w \in W$. For $d = 1, \dots, k$ set

$$(s_{i_d} \cdots s_{i_1}) \cdot \Lambda_\infty \;=\; \sum_{i=1}^n \Lambda_i \;-\; \sum_{i=1}^r \; l_i^d \, \alpha_i .$$

From [BGG] it follows that $l_{i_1}^1 > l_{i_1}$ and $l_{i_d}^d > l_{i_d}^{d-1}$ for $d = 2, \dots, k$.

Let $\boldsymbol{i} = (i_1, \dots, i_k)$, $1 \le i_j \le r$, be a sequence of integers. We consider the map $Y_{\boldsymbol{y}^0, \boldsymbol{i}} : (\mathbb{P}^1)^k \to \mathbb{P}(\mathbb{C}[x])^r$ introduced in Section 2.5 for special symmetric parameters $\boldsymbol{b} = (b_{i,j})$. Its image is denoted by $P_{\boldsymbol{y}^0, \boldsymbol{i}}$. The image of a point $(c_1, \dots, c_k) \in (\mathbb{P}^1)^k$ under this map is denoted by $\boldsymbol{y}^{k;\, c_1, \dots, c_k}$. We repeat the definition of $\boldsymbol{y}^{k;\, c_1, \dots, c_k}$ in terms convenient for our present purposes.

We assume that the tuple $\boldsymbol{y}^0$ is a tuple of monic polynomials. For $d = 1, \dots, k$ we define by induction on d a family of tuples of polynomials depending on parameters $c_1, \dots, c_d \in \mathbb{P}^1$. Namely, let $\tilde{y}_{i_1}$ be a polynomial satisfying equation

$$y_{i_1}^0(x+h)\, \tilde{y}_{i_1}(x) \;-\; y_{i_1}^0(x)\, \tilde{y}_{i_1}(x+h) \;=\; T_{i_1}(x) \prod_{j,\; j \ne i_1} (\, y_j^0(x + b_{i_1,j})\,)^{-a_{i_1,j}} \;.$$

We fix $\tilde{y}_{i_1}$ assuming that the coefficient of $x^{l_{i_1}}$ in $\tilde{y}_{i_1}$ is equal to zero. Set $\boldsymbol{y}^{1;\, c_1} = (y_1^{1;\, c_1}, \dots, y_r^{1;\, c_1}) \in \mathbb{P}(\mathbb{C}[x])^r$, where

$$y_{i_1}^{1;\, c_1}(x) \;=\; \tilde{y}_{i_1}(x) \;+\; c_1\, y_{i_1}^0(x) \qquad \text{and } y_j^{1;\, c_1}(x) \;=\; y_j^0(x) \text{ for } j \ne i_1 \;.$$

In particular, $\boldsymbol{y}^{1;\infty} = \boldsymbol{y}^0$ in $\mathbb{P}(\mathbb{C}[x])^r$.

Assume that the family $\boldsymbol{y}^{d-1;c_1,\dots,c_{d-1}} \in \mathbb{P}(\mathbb{C}[x])^r$ is already defined. Let $\tilde{y}_{i_d}^{d-1;c_1,\dots,c_{d-1}}$ be a polynomial satisfying equation

$$\begin{aligned} y_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x+h)\,\tilde{y}_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x) \;&-\; y_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x)\,\tilde{y}_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x+h) \\ &=\; T_{i_d}(x)\prod_{j,\ j\neq i_d}\left(\,y_j^{d-1;c_1,\dots,c_{d-1}}(x+b_{i_d,j})\,\right)^{-a_{i_d,j}}\,. \end{aligned}$$

We fix $\tilde{y}_{i_d}^{d-1;c_1,\dots,c_{d-1}}$ assuming that the coefficient of $x^{l_{i_d}^{d-1}}$ in $\tilde{y}_{i_d}^{d-1;c_1,\dots,c_{d-1}}$ is equal to zero. Set $\boldsymbol{y}^{d;c_1,\dots,c_d} = (y_1^{d;c_1,\dots,c_d},\dots,y_r^{d;c_1,\dots,c_d}) \in \mathbb{P}(\mathbb{C}[x])^r$, where

$$y_{i_d}^{d;c_1,\dots,c_d}(x) \;=\; \tilde{y}_{i_d}^{d-1;c_1,\dots,c_d}(x) \;+\; c_d\,y_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x)$$

and

$$y_j^{d;c_1,\dots,c_d}(x) \;=\; y_j^{d-1;c_1,\dots,c_{d-1}}(x) \quad \text{for} \quad j\neq i_d\,.$$

In particular, $\boldsymbol{y}^{d;c_1,\dots,c_{d-1},\infty} = \boldsymbol{y}^{d-1;c_1,\dots,c_{d-1}}$ in $\mathbb{P}(\mathbb{C}[x])^r$.

The d-th family is obtained from the $(d-1)$-st family by the generation procedure in the i_d-th direction, see Section 2.5. For any $(c_1,\dots,c_k)\in(\mathbb{P}^1)^k$ the tuple $\boldsymbol{y}^{k;c_1,\dots,c_k}$ lies in P.

For any $(c_1,\dots,c_k)\in\mathbb{C}^k$ and any $i\in\{1,\dots,r\}$, we have

$$\deg\ y_i^{k;c_1,\dots,c_k}(x) \;=\; l_i^w\,.$$

Set

$$P^{(i_1,\dots,i_k)} \;=\; \{\ \boldsymbol{y}^{k;c_1,\dots,c_k}\ |\ (c_1,\dots,c_k)\in\mathbb{C}^k\ \}\,.$$

For every $(c_1,\dots,c_k)\in(\mathbb{P}^1)^k$ we define a rational function $v_{c_1,\dots,c_k}:\mathbb{C}\to N_+$ by the formula

$$v_{c_1,\dots,c_k}: x\mapsto \exp\left(g_k(x;c_1,\dots,c_k)E_{i_k}\right)\dots\left(\exp\left(g_1(x;c_1)E_{i_1}\right)\right) \tag{22}$$

where

$$g_d(x;c_1,\dots,c_d) = \frac{T_{i_d}(x)\prod_{j,\ j\neq i_d}\left(\,y_j^{d-1;c_1,\dots,c_{d-1}}(x+b_{i_d,j})\,\right)^{-a_{i_d,j}}}{y_{i_d}^{d;c_1,\dots,c_d}(x)\ y_{i_d}^{d-1;c_1,\dots,c_{d-1}}(x)}$$

for $d=1,\dots,k$. In particular, if some of $c_1,\dots,c_k$ are equal to ∞, then the corresponding exponential factors in (22) must be replaced by the identity element $\mathrm{id}\in G$.

The function $v_{c_1,\dots,c_k}$ continuously depends on $(c_1,\dots,c_k)\in(\mathbb{P}^1)^k$. For any $(c_1,\dots,c_k)\in(\mathbb{P}^1)^k$ the pair $(D^{v_{c_1,\dots,c_k}}, v_{c_1,\dots,c_k})$ belongs to $\widehat{\mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}}$.

Let $x_0\in\mathbb{C}$ be a regular point. Consider the map

$$\phi:\mathbb{C}^k\to G/B_-,\qquad (c_1,\dots,c_k)\mapsto (v_{c_1,\dots,c_k}(x_0))^{-1}B_-\,.$$

Proposition 6.8. *The image of the map π is $B_{w^{-1}}$.*

Proof of Proposition 6.8. For any $(c_1, \dots, c_k) \in (\mathbb{P}^1)^k$ consider the rational section

$$S_{(c_1,\dots,c_k)} \ : \ x \ \mapsto \ ((v_{c_1,\dots,c_k}(x))^{-1} B_-) \times x \tag{23}$$

of the bundle p'. In particular, if some of $c_1, \dots, c_k$ are equal to ∞, then the corresponding exponential factors in (22) must be replaced by the identity element $\mathrm{id} \in G$. This section is horizontal with respect to the connection $\nabla'_{\boldsymbol{y}^0}$ and continuously depends on $(c_1, \dots, c_k) \in (\mathbb{P}^1)^k$. This means that

$$\phi(\mathbb{C}^k) \ \subset \ B_{w^{-1}} \ ,$$

see Corollary 6.2, and if some of $c_1, \dots, c_k$ are equal to ∞, then $S_{(c_1,\dots,c_k)}(x) \notin B_{w^{-1}}$, see Corollary 6.3. It remains to show that every point in $B_{w^{-1}}$ is the limit of points of the form $S_{(c_1,\dots,c_k)}(x)$. But that statement follows from Corollary 6.2 and

Lemma 6.9. *Assume that $x_0 \in \mathbb{C}$ is such that $T_i(x_0) \neq 0$, $y_i^0(x_0) \neq 0$, for $i = 1, \dots, r$. Assume that $x_0 \in \mathbb{C}$ is such that $y_j^0(x_0 + b_{i,j}) \neq 0$ for all $i \neq j$. Then there exists a proper algebraic subset $K \subset (\mathbb{C} - 0)^k$ with the following property. For any $(c_1^1, \dots, c_k^1) \in (\mathbb{C} - 0)^k - K$ there exists a unique $(c_1^2, \dots, c_k^2) \in \mathbb{C}^k$ such that*

$$(c_1^1, \dots, c_k^1) \ = \ (g_1(x_0; c_1^2), \dots, g_k(x_0; c_1^2, \dots, c_k^2)) \ .$$

$\square$

The proposition is proved. $\square$

Theorem 6.1 is a direct corollary of Proposition 6.8.

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